EEE8058 Nonlinear Systems / Final Exam

January 10, 2018

1. (20 pts) Consider the system $\dot{x} = -y + \mu x + xy^2$, $\dot{y} = x + \mu y - x^2$.

   a) (10 pts) Rewrite the system in polar coordinates.

   b) (10 pts) Show that if $r \ll 1$, then $\dot{\theta} \approx 1$ and $\dot{r} \approx \mu r + \frac{1}{8} r^3 + \cdots$, where the terms ommited are oscillatory and have essentially zero time-average around one cycle.

2. (25 pts) Consider the system

$$\dot{x} = x[x(1-x) - y], \quad \dot{y} = y(x - a)$$

   where $x \geq 0$ is the dimensionless population of the prey, $y \geq 0$ is the dimensionless population of the predator, and $a \geq 0$ is a control parameter.

   a) (5 pts) Sketch the nullclines in the first quadrant $x, y \geq 0$.

   b) (5 pts) Show that the equilibrium points are $(0,0), (1,0)$ and $(a, a - a^2)$ and classify them.

   c) (5 pts) Sketch the phase portrait for $a > 1$ and show that the predator go exinct.

   d) (10 pts) Show that the Hopf bifurcation occurs at $a_c = \frac{1}{2}$. Is it subcritical or supercritical? Justify your answer.

3. (25 pts) Prove non-existence of limit cycles for the system

$$\dot{x} = y^2 - x, \quad \dot{y} = -y^3 + 2xy + 2y$$

   by

   a) (5 pts) using index theory,

   b) (5 pts) showing that it is a gradient system, and

   c) (15 pts) using Dulac's criterion.