EEE8058 Nonlinear Systems / Homework 1

Due: October 3, 2018

1. Analyze the following nonlinear systems graphically by i) sketching the vector fields on the real line, ii) finding all the equilibrium points, iii) investigating their stability and iv) sketching $x(t)$ for different initial conditions.

    a) $\dot{x} = e^{-x} \sin x$

    b) $\dot{x} = 1 + \frac{1}{2} \cos x$

    c) $\dot{x} = 1 - 2 \cos x$

    d) $\dot{x} = e^{x} - \cos x$

2. Consider the chemical reaction model $A + X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2X$ in which one molecule of X combines with one molecule of A to form two molecules of X. This means that the chemical X stimulates its own production, a process called autocatalysis. This positive feedback process leads to a chain reaction, which eventually is limited by a back reaction in which $2X$ returns to $A + X$.

    According to law of mass action of chemical kinetics, the rate of an elementary reaction is proportional to the product of the concentrations of the reactants. We denote the concentrations by lowercase letters $x = [X]$ and $a = [A]$. Assume that there's an enormous surplus of chemical A so that its concentration a can be regarded as constant. Then the equation for the kinetics of x is

    $$\dot{x} = k_1 a x - k_{-1} x^2$$

    where k_1 and k_{-1} are positive parameters called rate constants.

    a) Find all equilibrium points of the equation and classify their stability.

    b) Sketch the graph of $x(t)$ for various initial values x_0.

3. The growth of cancerous tumors can be modeled by the Gompertz law $\dot{N} = -aN \ln(bN)$ where $N(t)$ is proportional to the number of cells in the tumor and $a, b > 0$ are parameters. Sketch the vector field and then graph $N(t)$ for various initial values.

4. In statistical mechanics, the phenomenon of critical slowing down is a signature of a second order phase transition. At the transition, the system relaxes to equilibrium much more slowly than usual. Here's a mathematical version of this effect:

    a) Obtain the analytic solution to $\dot{x} = -x^3$ for an arbitrary initial condition. Show that $x(t) \to 0$ as $t \to \infty$, but that the decay is not exponential.

    b) In order to get some intuition about the slowness of the decay, make a numerically accurate plot of the solution for the initial condition $x_0 = 10$ for $0 \leq t \leq 10$. Then on the same graph, plot the solution to $\dot{x} = -x$ for the same initial condition.