EEE8058 Nonlinear Systems / Homework 2

Due: October 10, 2018

1. For the system with nonlinear dynamics given by the equation $\dot{x} = r + x - \ln(1+x)$, sketch all the qualitatively different vector fields that occur as r is varied. Find the critical value of r at which a saddle-node bifurcation occurs. Finally, sketch the bifurcation diagram of equilibrium points x^* versus r.

2. For the system with nonlinear dynamics given by the equation $\dot{x} = rx - \ln(1+x)$, sketch all the qualitatively different vector fields that occur as r is varied. Find the critical value of r at which a transcritical bifurcation occurs. Finally, sketch the bifurcation diagram of equilibrium points x^* versus r.

3. For the system with nonlinear dynamics given by the equation $\dot{x} = rx - \dfrac{x}{1+x}$, find the values of r at which bifurcations occur, and determine the type of the bifurcation (saddle-node, transcritical, supercritical pitchfork or subcritical pitchfork). Furthermore, sketch the bifurcation diagram of equilibrium points x^* versus r.

4. Consider the system $\dot{x} = rx + x^3 - x^5$ which exhibits a subcritical pitchfork bifurcation.

   a) Find the algebraic expressions for all the equilibrium points as x varies.

   b) Sketch the vector field as r varies. Be sure to indicate all the equilibrium points and their stability.

   c) Calculate r_s, the parameter value at which the nonzero equilibrium points are born in a saddle node bifurcation.

5. Zebra stripes and butterfly wing patterns are two of the most spectacular examples of biological pattern formation. Explaining the development of these patterns is one of the most outstanding problems of biology.

   As one ingredient in a model of pattern formation, Lewis *et. al.* considered a simple example of a biochemical switch, in which a gene G is activated by a biochemical signal substance S. For example, the gene may normally be inactive but can be *switched on* to produce a pigment or other gene product when the concentration of S exceeds a certain threshold. Let $g(t)$ denote the concentration of the gene product and assume that the concentration s_0 of S is fixed. The model is

   $$\dot{g} = k_1 s_0 - k_2 g + \frac{k_3 g^2}{k_4^2 + g^2}$$

   where the k's are positive constants. The production of g is stimulated by s_0 at a rate k_1 and by an autocatalytic or positive feedback process (the nonlinear term). There is also a linear degradation of g at a rate k_2.

a) Show that the system can be put in the dimensionless form

$$\frac{dx}{d\tau} = s - rx + \frac{x^2}{1+x^2}$$

where $r > 0$ and $s \geq 0$ are dimensionless groups.

b) Show that if $s = 0$, there are two positive equilibrium points x^* if $r < r_c$ where r_c is to be determined.

c) Assume that initially there is no gene product, i.e., $g(0) = 0$, and suppose s is slowly increased from zero (the activating signal is turned on); what happens to $g(t)$? What happens if s then goes back to zero? Does the gene turn off again?

d) Find parametric equations for the bifurcation curves in (r, s) space and classify the bifurcations that occur.

e) Use a computer program to give a quantitatively accurate plot of the stability diagram in (r, s) space.